

# ENLARGEABILITY AND INDEX THEORY: INFINITE COVERS

BERNHARD HANKE AND THOMAS SCHICK

**ABSTRACT.** In [5] we showed non-vanishing of the universal index elements in the  $K$ -theory of the maximal  $C^*$ -algebras of the fundamental groups of enlargeable spin manifolds. The underlying notion of enlargeability was the one from [3], involving contracting maps defined on finite covers of the given manifolds. In the paper at hand, we weaken this assumption to the one in [4] where infinite covers are allowed. The new idea is the construction of a geometrically given  $C^*$ -algebra with trace which encodes the information given by these infinite covers. Along the way we obtain an easy proof of a relative index theorem relevant in this context.

## 1. OVERVIEW OF RESULTS

This paper is a sequel to [5]. We quickly review the relevant mathematics and the motivation for our research. Since the work of Lichnerowicz it has been known that index theory can be used to construct computable obstructions to the existence of positive scalar curvature metrics on closed spin manifolds  $M^n$ : If such a metric exists on  $M$ , then  $\hat{A}(M)$ , a characteristic number of  $M$  involving its Pontryagin classes, vanishes. However, the converse is not true: In [3] it was shown (again by index theoretic methods) that also enlargeability (see the definition below) of  $M$  is an obstruction to the existence of a positive scalar curvature metric. Furthermore, it is well known that there are enlargeable manifolds with vanishing  $\hat{A}$ -genus (for example tori). The  $\hat{A}$ -obstruction was gradually refined and in its most sophisticated form constructed by Rosenberg [7] as an element

$$\alpha_{max}^{\mathbb{R}}(M) \in KO_n(C_{max, \mathbb{R}}^* \pi_1(M))$$

in the real  $K$ -theory of the maximal  $C^*$ -algebra of  $\pi_1(M)$ . It is often assumed that this obstruction subsumes all other index theoretic obstructions against positive scalar curvature metrics on  $M$ . Results

---

We thank S. Stolz and A. Thom for useful conversations regarding the research in this paper. Both authors are members of the DFG emphasis programme “Globale Differentialgeometrie” whose support is gratefully acknowledged.

of Stolz [10] imply that this is indeed the case, if the Baum-Connes conjecture holds for  $\pi_1(M)$ . Note that the emphasis here is on index theoretic obstructions. The example of [8] shows that in general there are other obstructions which can not be detected by  $\alpha_{max}^{\mathbb{R}}(M)$ .

Guided by these facts, it is natural to ask whether non-vanishing of  $\alpha_{max}^{\mathbb{R}}(M)$  for enlargeable  $M$  can be shown independently of the Baum-Connes conjecture. In our paper [5] this was achieved for manifolds that are enlargeable in the sense of [3]. It is the purpose of this note to extend this result to manifolds that are enlargeable in the more general sense of [4].

Let us start with the relevant definition introduced in [4].

**Definition 1.1.** We call a connected closed oriented manifold  $M^n$  *enlargeable* if the following holds: Fix some Riemannian metric  $g$  on  $M$ . Then, for all  $\epsilon > 0$ , there is a connected covering  $\overline{M} \rightarrow M$  (which may be finite or infinite) and an  $\epsilon$ -contracting map  $(\overline{M}, \overline{g}) \rightarrow (S^n, g_0)$  which is constant outside a compact subset of  $\overline{M}$  and of nonzero degree. Here,  $\overline{g}$  is induced by  $g$  and  $g_0$  is the standard metric on  $S^n$ .

The manifold  $M$  is called *area-enlargeable* if in the above definition  $\epsilon$ -contracting is replaced by  $\epsilon$ -area-contracting (cf. [5], Def. 1.1).

If we require in addition the covers  $\overline{M} \rightarrow M$  to be finite, this coincides with Definition 1.1 in [5] which goes back to [3]. It is an easy consequence of Hadamard's theorem that  $M$  is enlargeable, if it admits a Riemannian metric of nonnegative sectional curvature. Moreover, the connected sum of an enlargeable manifold with an arbitrary connected closed oriented manifold is again enlargeable.

From now on the notion of (area-)enlargeability refers to Definition 1.1. In contrast, manifolds sharing the more restrictive property used in [5] will be called *compactly* (area-)enlargeable.

It is obvious that each enlargeable manifold is area-enlargeable, but not known if the converse holds.

Because we are mainly interested in explaining the method, we confine ourselves to indicating a proof of the following analogue of one of the main results of [5].

**Theorem 1.2.** *Let  $M$  be an enlargeable or area-enlargeable spin manifold of dimension  $n$ . Then*

$$\alpha_{max}(M) \neq 0 \in K_n(C_{max}^* \pi_1(M)).$$

Here and in the rest of this paper we use complex  $K$ -theory. The statement of Theorem 1.2 implies a similar non-vanishing statement for  $\alpha_{max}^{\mathbb{R}}(M)$ .

We mention the following related result. The proof of the corresponding statement in [5] applies without change.

**Theorem 1.3.** *Let  $M$  be an area-enlargeable manifold of dimension  $n$  and let*

$$f : M \rightarrow B\pi_1(M)$$

*be the classifying map of the universal cover of  $M$ . Then the image of the fundamental class of  $M$  under the induced map in homology*

$$f_* : H_n(M; \mathbb{Q}) \rightarrow H_n(B\pi_1(M); \mathbb{Q})$$

*is different from zero.*

The starting point for the discussion in [5] is the observation (inspired by the work of Gromov and Lawson) that on an even dimensional compactly area-enlargeable manifold, one can construct an almost flat vector bundle whose total Chern class is different from zero exactly in degrees 0 and  $n$  (where  $\dim M = 2n$ ). By definition (cf. [5], Section 2), this means that we have a sequence  $(E_i)_{i \in \mathbb{N}}$  of finite dimensional unitary vector bundles on  $M$  whose total Chern classes have the described property and that enjoy the following asymptotic flatness property: Each bundle  $E_i$  is equipped with a metric connection  $\nabla_i$  so that the associated curvature 2-forms ( $d_i := \dim E_i$ )

$$\Omega_i \in \Omega^2(M; \mathfrak{u}(d_i))$$

vanish asymptotically in the sense that

$$\lim_{i \rightarrow \infty} \|\Omega_i\| = 0.$$

Here we use the maximum norm on the unit sphere bundle in  $\Lambda^2 M$  and the operator norm on  $\mathfrak{u}(d_i) \subset \text{Mat}(d_i) = \mathbb{C}^{d_i \times d_i}$ .

The main contribution of the paper at hand is the proof of a similar statement under the assumption that  $M$  is area-enlargeable in the sense of Definition 1.1. The main difference is that now the constituents of the almost flat bundle may be infinite dimensional.

**Definition 1.4.** Let  $M$  be a closed smooth Riemannian manifold. An *almost flat Hilbert-module bundle* consists of a sequence  $(C_i)_{i \in \mathbb{N}}$  of complex  $C^*$ -algebras (which may be unital or non-unital) and a sequence  $(F_i, \nabla_i)$  of Hilbert  $C_i$ -module bundles over  $M$  whose fibres are  $C_i$ -isomorphic to projective right  $C_i$ -modules of the form  $q_i C_i$ , where  $q_i \in C_i$  is a projection. Furthermore we require that each  $F_i$  is equipped with a metric  $C_i$ -linear connection

$$\nabla_i : \Gamma(F_i) \rightarrow \Gamma(T^*M \otimes F_i)$$

so that the associated curvature forms

$$\Omega_i \in \Omega^2(M; \text{End}(F_i))$$

tend to zero in the sense that

$$\lim_{i \rightarrow \infty} \|\Omega_i\| = 0.$$

For a detailed discussion of Hilbert module bundles, we refer to [9].

The essential step in our discussion is the following technical result. We denote by  $\mathbb{K}$  the  $C^*$ -algebra of compact operators on the Hilbert space  $l^2(\mathbb{N})$ .

**Proposition 1.5.** *Let  $M^{2n}$  be an even dimensional area-enlargeable spin manifold and let  $i \in \mathbb{N}$  be a positive natural number. Then there is a  $C^*$ -algebra  $C_i$  and a Hilbert  $C_i$ -module bundle  $F_i \rightarrow M$  with connection  $\nabla_i$  with the following properties: The curvature  $\Omega_i$  of  $F_i$  satisfies*

$$\|\Omega_i\| \leq \frac{1}{i}C$$

where  $C$  is a constant depending only on  $n$ . Furthermore, there is a split extension of the form

$$0 \rightarrow \mathbb{K} \rightarrow C_i \rightarrow \Gamma_i \rightarrow 0$$

with a certain  $C^*$ -algebra  $\Gamma_i$ . In particular, each  $K_0(C_i)$  canonically splits off a  $\mathbb{Z} = K_0(\mathbb{K})$  summand. Let  $a_i \in K_0(C_i)$  denote the index of the spin Dirac operator on  $M$  twisted with  $F_i$ . Then the  $\mathbb{Z} = K_0(\mathbb{K})$ -component  $z_i$  of  $a_i$  is different from 0.

We make some remarks concerning the idea behind this statement. Using the area-enlargeability of  $M$ , there is a cover  $\overline{M} \rightarrow M$  and a unitary bundle  $F \rightarrow \overline{M}$  of finite dimension  $d$  whose curvature norm is bounded by  $\frac{1}{i}C$ . Moreover, if  $\overline{M}$  is non-compact, this bundle is trivial outside a certain compact subset of  $\overline{M}$ . The bundle  $F$  has the property that the (relative) index of the Dirac operator on  $\overline{M}$  twisted with the virtual bundle  $F - \underline{\mathbb{C}}^d$  is different from zero. The construction of  $F$  is explained in [4] and will be reviewed in Section 2 below. Proposition 1.5 claims that this relative index (which will be equal to  $z_i$ ) can be read off in a canonical way from the index of a twisted Dirac operator on  $M$ . This argument is carried out in our previous paper [5], if  $\overline{M}$  is compact (i.e. if the cover  $\overline{M} \rightarrow M$  is finite). The purpose of Proposition 1.5 is a generalization of this construction to the case of non-compact  $\overline{M}$ .

Before proving Proposition 1.5 in the next section, we explain how Theorem 1.2 follows.

By an easy suspension argument (cf. the proof of Theorem 4.2. in [5]) we may restrict attention to even dimensional  $M$ .

Let  $A$  denote the complex  $C^*$ -algebra of norm bounded sequences

$$(a_i)_{i \in \{1,2,\dots\}} \in \prod_{i=1}^{\infty} C_i$$

and for  $i \in \mathbb{N}$ , we denote by  $A_i \subset A$  the subalgebra where all but the  $i$ th entry vanish. This subalgebra can be identified with  $C_i$ . Let

$$q := (q_i) \in A$$

be the projection consisting of the projections  $q_i$ . We now have an analogue of [5], Theorem 2.1: The bundles  $F_i$  together with the connections  $\nabla_i$  can be assembled to a smooth Hilbert  $A$ -module bundle  $V$  together with an  $A$ -linear metric connection

$$\nabla^V : \Gamma(V) \rightarrow \Gamma(T^*M \otimes V)$$

such that for each  $i \in \mathbb{N}$ , the subbundle  $V_i := V \cdot A_i$  is isomorphic to  $F_i$  as an  $A_i$ -Hilbert module bundle, the connection  $\nabla^V$  preserves the subbundles  $V_i$  and the norms of the induced connections  $\nabla_i^V$  on  $V_i$  tend to zero. The proof of this statement is similar to the proof of [5, Theorem 2.1] and will be omitted.

We now argue as in Section 3 of [5] and define

$$A' = \overline{\bigoplus_{i=1}^{\infty} A_i} \subset A$$

as the closed two sided ideal consisting of sequences converging to 0. Let

$$Q := A/A'$$

denote the quotient  $C^*$ -algebra. Similarly, we define  $A_\Gamma := \prod_{i \in \mathbb{N}} \Gamma_i$  with ideal  $A'_\Gamma := \bigoplus_{i \in \mathbb{N}} \Gamma_i$  and quotient  $Q_\Gamma := A_\Gamma/A'_\Gamma$ . The bundle

$$W := V/(V \cdot A') = V \times_A Q \rightarrow M$$

is a smooth Hilbert  $Q$ -module bundle with fibre  $\bar{q}Q$  where  $\bar{q} \in Q$  is the image of  $q$ . The connection  $\nabla^V$  induces a  $Q$ -linear metric connection on  $W$ . The crucial point is that this connection is flat (cf. [5, Proposition 4.3]). After fixing a basepoint in  $x \in M$  and an isomorphism  $W_x \cong \bar{q}Q$ , we hence get a holonomy representation

$$\pi_1(M, x) \rightarrow \text{Hom}_Q(\bar{q}Q, \bar{q}Q) = \bar{q}Q\bar{q}$$

with values in the unitaries of  $\bar{q}Q\bar{q}$ . Using the universal property of  $C_{max}^*\pi_1(M)$ , we obtain a  $C^*$ -algebra homomorphism

$$\phi : C_{max}^*\pi_1(M) \rightarrow \bar{q}Q\bar{q} \rightarrow Q$$

and hence an induced map in  $K$ -theory

$$\phi_* = K_0(C_{max}^* \pi_1(M)) \rightarrow K_0(Q).$$

One now shows that the image of  $\alpha_{max}(M)$  under this map is different from 0. For that purpose we need some information on the  $K$ -theory of  $Q$ . Consider the following commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus_{i \in \mathbb{N}} \mathbb{K} & \longrightarrow & A' & \longrightarrow & A'_\Gamma \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_{i \in \mathbb{N}} \mathbb{K} & \longrightarrow & A & \longrightarrow & A_\Gamma \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (\prod_{i \in \mathbb{N}} \mathbb{K}) / (\bigoplus_{i \in \mathbb{N}} \mathbb{K}) & \longrightarrow & Q & \longrightarrow & Q_\Gamma \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where the horizontal rows are split exact (and the splits also make the diagram commutative) and the vertical columns are exact. In [5, Proposition 3.5], we calculated the  $K$ -theory of the left vertical column. It follows that the group  $K_0(Q)$  splits off a summand

$$(\prod_{i=1}^{\infty} \mathbb{Z}) / (\bigoplus_{i=1}^{\infty} \mathbb{Z})$$

which is the image of a corresponding summand  $\prod_{i \in \mathbb{N}} \mathbb{Z}$  in  $K_0(A)$  under the map induced by the projection. Moreover, as in [5] one now checks that the component of  $\phi_*(\alpha_{max}(M))$  in this summand is represented by the sequence

$$(z_1, z_2, \dots)$$

and is therefore different from 0. This implies that  $\phi_*(\alpha_{max}(M)) \neq 0$  and hence  $\alpha_{max}(M) \neq 0$  which completes the proof of Theorem 1.2.

## 2. PROOF OF PROPOSITION 1.5

From now on, fix  $i \in \mathbb{N}$ . Since  $M^{2n}$  is area-enlargeable, there is a covering

$$f : \overline{M} \rightarrow M$$

together with an  $\frac{1}{i}$ -area-contracting map

$$\psi : \overline{M} \rightarrow S^{2n}$$

which is constant outside a certain compact subset  $K \subset \overline{M}$  (which will be fixed from now on) and of nonzero degree. If  $\overline{M}$  is compact, we set  $K = \overline{M}$ .

We fix a finite dimensional unitary vector bundle

$$E \rightarrow S^{2n}$$

whose  $n$ -th Chern class is different from zero. This is possible because the Chern character is a rational isomorphism. Let  $d$  be the complex dimension of  $E$ . We fix a unitary connection on  $E$ .

Now let

$$F := \psi^*(E) \rightarrow \overline{M}$$

be the pull back bundle with its induced unitary structure and connection  $\nabla_F$ . Let

$$\Omega_F \in \Omega^2(\overline{M}; \mathfrak{u}(d)), \quad \Omega_E \in \Omega^2(S^n; \mathfrak{u}(d))$$

be the curvature forms of  $F$  and  $E$ . We set

$$C := \|\Omega_E\|.$$

Because  $\psi$  is  $\frac{1}{i}$ -area contracting, we have

$$\|\Omega_F\| \leq \frac{1}{i} C.$$

We would like to use the bundle  $F$  to construct a bundle on  $M$  with connection whose curvature norm is related to  $C$  in the same way. We think of the fibre over  $x \in M$  as the direct sum of all the fibres  $F_y$  where  $y \in f^{-1}(x)$ . If the degree of  $\overline{M} \rightarrow M$  is infinite then this “wrapped up” bundle will have infinite dimension and for arriving at the statement of Proposition 1.5 we must keep control of the “structure group” of this infinite dimensional bundle.

Because  $\psi$  is constant outside  $K$ , there is a unitary trivialization

$$F|_{\overline{M} \setminus K} \cong (\overline{M} \setminus K) \times \mathbb{C}^d.$$

such that with respect to this trivialization, the connection 1-form of the connection on  $F$  vanishes.

Moreover, there is a finite cover

$$U_1, \dots, U_k$$

of  $M$  consisting of open connected subsets such that the restriction of  $F$  to  $f^{-1}(U_\kappa)$  is trivial for all  $\kappa \in \{1, 2, \dots, k\}$ . By choosing the members of this cover sufficiently small, we can furthermore assume that for all  $\kappa, \lambda \in \{1, \dots, k\}$  and for each component  $V \subset f^{-1}(U_\kappa)$

there is at most one component  $W \subset f^{-1}(U_\lambda)$  with  $V \cap W \neq \emptyset$  and that each  $U_\kappa$  is evenly covered by the universal cover

$$\pi : \widetilde{M} \rightarrow M$$

i.e. each component of  $\pi^{-1}(U_\kappa)$  is mapped diffeomorphically to  $U_\kappa$ .

Pick a basepoint in  $\overline{M}$  and the induced base point in  $M$ . Now the fundamental group

$$H := \pi_1(\overline{M})$$

acts freely and transitively from the right on  $\widetilde{M}$ . For each  $\kappa$  this induces a parameterization of the set of components of  $f^{-1}(U_\kappa)$  by  $G/H = \{gH \mid g \in G\}$ , where

$$G := \pi_1(M).$$

However this parameterization is not canonical, but depends on singling out a preferred component of  $f^{-1}(U_\kappa)$ . We assume that this has been done. For each  $\kappa$  and each component  $V \subset f^{-1}(U_\kappa)$ , we choose a unitary trivialization of  $F|_V$ . If  $V \cap K = \emptyset$ , we take the trivialization which was already chosen before.

Let

$$\mathcal{H} := l^2(G/H) \otimes \mathbb{C}^d$$

be the Hilbert space of square summable  $G/H$ -families of vectors in  $\mathbb{C}^d$ . This space is finite dimensional, if  $\overline{M}$  is compact, and a separable infinite dimensional Hilbert space, if  $\overline{M}$  is not compact ( $G/H$  is then countably infinite). For each  $\kappa, \lambda \in \{1, 2, \dots, k\}$ , we obtain a transition function

$$\alpha_{\kappa\lambda} : (U_\kappa \cap U_\lambda) \rightarrow \mathcal{B}(\mathcal{H})$$

describing the passage from the trivialization of  $F$  over  $f^{-1}(U_\kappa)$  to the trivialization over  $f^{-1}(U_\lambda)$ . Because  $F$  is unitary, the image of  $\alpha_{\kappa\lambda}$  is contained in the group of isometries of  $\mathcal{H}$ .

The group of permutations of  $G/H$  acts isometrically on  $\mathcal{H}$  by precomposition in  $l^2(G/H)$ . Let

$$C_S \subset \mathcal{B}(\mathcal{H})$$

be the norm closure of the algebra generated by these isometries. Furthermore, let

$$C_T \subset \mathcal{B}(\mathcal{H})$$

be the  $C^*$ -algebra generated by “elementary” linear maps

$$\gamma_0 \otimes \mathbb{C}^d \rightarrow \mu_0 \otimes \mathbb{C}^d$$



extended by zero on the orthogonal complement of  $\gamma_0 \otimes \mathbb{C}^d \subset l^2(G/H) \otimes \mathbb{C}^d$ , where  $\gamma_0, \mu_0 \in G/H$  are arbitrary (considered as elements in  $l^2(G/H)$ ). Let

$$C_{S,T} \subset \mathcal{B}(\mathcal{H})$$

be generated by  $C_S$  and  $C_T$ . Note that  $C_T = C_{S,T}$ , if  $\overline{M}$  is compact, and that  $C_T$  is a proper subalgebra of  $C_{S,T}$  which can be identified with  $\mathbb{K}$ , if  $\overline{M}$  is not compact. Furthermore,  $C_T$  is a two sided ideal in  $C_{T,S}$ .

We now set

$$C_i := \{ (c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in C_T \}.$$

An easy calculation shows that this is indeed a  $C^*$ -algebra. It is equipped with a unit.

*Remark 2.1.* The construction of this algebra is inspired by the discussion in John Roe's article [6].

The projection  $C_{S,T} \times C_{S,T} \rightarrow C_{S,T}$  onto the second factor induces a split exact sequence

$$0 \rightarrow C_T \rightarrow C_i \rightarrow C_{S,T} \rightarrow 0,$$

the splitting being induced by the diagonal embedding

$$C_{S,T} \rightarrow C_{S,T} \times C_{S,T}.$$

Because the cover  $(U_\kappa)$  is sufficiently small (in the sense explained before), for each transition function  $\alpha_{\kappa\lambda}$  there is a uniquely determined permutation  $\beta_{\kappa\lambda}$  of  $G/H$  such that the image of the composition

$$(\beta_{\kappa\lambda})^{-1} \circ \alpha_{\kappa\lambda}$$

is contained in a  $G/H$ -family of unitary maps  $\mathbb{C}^d \rightarrow \mathbb{C}^d$  all but finitely many of which are equal to the identity. This gives rise to relative transition functions

$$\begin{aligned} \rho_{\kappa\lambda} : U_\kappa \cap U_\lambda &\rightarrow C_{S,T} \times C_{S,T}, \\ x &\mapsto (\alpha_{\kappa,\lambda}(x), \beta_{\kappa,\lambda}(x)). \end{aligned}$$

Note that  $\beta_{\kappa,\lambda}$  is locally constant, and that these transition functions satisfy the cocycle condition. The following lemma is immediate.

**Lemma 2.2.** *Each relative transition function  $\rho_{\kappa\lambda}$  is smooth and has image contained in the unitaries of  $C_i$ .*

We now set

$$F_i := \prod_{j=1, \dots, k} (j, U_j \times C_i) / ((\kappa, x, v) \sim (\lambda, x, \rho_{\kappa\lambda}(x) \cdot v)).$$

By Lemma 2.2 this is a smooth Hilbert  $C_i$ -module bundle over  $M$ . Its fibres are isomorphic to  $C_i$  and are equipped with the  $C_i$ -valued inner product induced by

$$C_i \times C_i \rightarrow C_i, (x, y) \mapsto x^* \cdot y.$$

Over each subset  $U_\kappa \subset M$ , a section of  $F_i|_{U_\kappa}$  defines a  $G/H$ -family of sections of the trivial bundle  $\underline{\mathbb{C}}^d \rightarrow U_\kappa$  (corresponding to the components of  $f^{-1}(U_\kappa)$ ). Furthermore, the connection on  $F$  induces a  $G/H$ -family of connections on  $\underline{\mathbb{C}}^d$  all but finitely many of which are trivial. The last property is implied by the choice of trivialization of  $F|_{\overline{M} \setminus K}$ . These considerations show that we get an induced  $C_i$ -linear metric connection

$$\nabla_i : \Gamma(F_i) \rightarrow \Gamma(T^*M \otimes F_i).$$

In the above trivialization of  $F_i$ , this is given by  $C_i \cap (C_T \times 0)$ -valued 1-forms on each  $U_j$ . Note that over each  $U_\kappa \subset M$ , the curvature

$$\Omega^2(U_\kappa; \text{End}_{C_i}(F_i)) = \Omega^2(U_\kappa; C_i)$$

can be viewed as a form in  $\Omega^2(U_\kappa)$  with values in a  $G/H$ -family of matrices  $\mathbf{u}(d)$ , all but finitely many of which are equal to zero. Note also that the norm  $\|\Omega_{F_i}\|$  of the curvature of  $\nabla_i$  is equal to  $\|\Omega_F\|$ . In particular, it satisfies the inequality

$$\|\Omega_{F_i}\| \leq \frac{1}{i} C$$

which we have been aiming at.

If  $\overline{M}$  is not compact, then  $C_T$  can be identified with  $\mathbb{K}$  and the algebra  $C_i$  has the form described in Proposition 1.5. However, if  $\overline{M}$  is compact, then  $C_T$  is a finite dimensional matrix algebra. In this case we replace the Hilbert space  $\mathcal{H}$  by its stabilization  $\mathcal{H} \otimes l^2(\mathbb{N})$  and correspondingly replace  $C_T$ ,  $C_S$ ,  $C_{S,T}$  and  $C_i$  by  $C_T \otimes \mathbb{K}$ , etc. The inclusion  $\mathbb{C} \hookrightarrow l^2(\mathbb{N})$  onto the first factor induces canonical inclusions  $C_T \hookrightarrow C_T \otimes \mathbb{K}$ , etc. In particular, if

$$p : l^2(\mathbb{N}) \rightarrow \mathbb{C} \subset l^2(\mathbb{N})$$

denotes the projection onto the first factor, then

$$q_i := 1 \otimes p \in C_i \otimes \mathbb{K}$$

is a projection and the transition functions  $\rho_{\kappa\lambda}$  have images that are contained in the unitaries of  $q_i(C_i \otimes \mathbb{K})$ . Replacing the typical fibre  $C_i$  by  $q_i(C_i \otimes \mathbb{K})$  in the definition of the bundle  $F_i$ , we then arrive at the statement of Proposition 1.5, if we set  $q_i := 1$  in the case of non-compact  $\overline{M}$ . From now on we work with these modified objects (without change of notation), if  $\overline{M}$  is compact.

*Remark 2.3.* This stabilization process is done only for convenience, because we want to make sure that  $C_i$  splits off a summand  $\mathbb{K}$  independently of  $\overline{M}$  being compact or not. This makes sure that  $K_0(\prod C_i)$  splits off a summand  $\prod \mathbb{Z}$ .

It remains to calculate the component in the  $\mathbb{Z}$ -summand of the index of the spin Dirac operator on  $M$  twisted with the Hilbert  $C_i$ -module bundle  $F_i$  with connection.

We use the Chern-Weil calculus developed in [9]. Let

$$\mathcal{T} \subset C_T = \mathbb{K}$$

be the subalgebra of trace class operators and set

$$D := \{ (c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in \mathcal{T} \} \subset C_i.$$

Note that the projection  $q_i$  is contained in  $D$ . We get again a split exact sequence

$$0 \rightarrow \mathcal{T} \rightarrow D \rightarrow C_{S,T} \rightarrow 0.$$

The algebra  $\mathcal{T}$  is a two sided ideal in  $D$ . The sum of the trace-norm on  $\mathcal{T}$  and the  $C^*$ -algebra norm on  $C_{S,T}$  induce a Banach algebra structure on  $D$  so that the inclusion

$$D \rightarrow C_i$$

is continuous. These facts follow from standard properties of the trace-norm (see [2, I.8.5.6]) and the triangle inequality for the  $C^*$ -algebra norm on  $C_i$ . Consider the map

$$\begin{aligned} \tau : D &\rightarrow \mathbb{C} \\ (c_1, c_2) &\rightarrow \operatorname{tr}(c_1 - c_2), \end{aligned}$$

using the canonical trace on  $\mathcal{T}$ . It is easily verified that  $\tau$  defines a trace on  $D$  and that

$$\tau(c) = \operatorname{tr}(c_{\mathcal{T}}),$$

where  $c_{\mathcal{T}}$  is the projection of  $c$  onto the summand  $\mathcal{T}$  in  $D = \mathcal{T} \oplus C_{S,T}$ .

In order to apply the results from [9], we need the following lemma.

**Lemma 2.4.**  *$D$  is a dense subalgebra of  $C_i$  and closed under holomorphic functional calculus in  $C_i$ .*

*Proof.* The first assertion is obvious. For the second assertion we need to show that if  $c$  is an element in the matrix algebra  $M_k(D)$  and  $f$  is a holomorphic function defined in a neighbourhood of the spectrum of  $c$  regarded as an element in  $M_k(C_i)$ , then  $f(c) \in M_k(D)$ . By the construction of holomorphic functional calculus via the Cauchy integral formula (recall that  $D_i$  is a Banach algebra), it is enough to show that

each element in  $D$  which is invertible in  $C_i$  is also invertible in  $D$  (and the same for matrices).

Let

$$x = x_1 + x_2 \in \mathcal{T} \oplus C_{S,T} = D$$

be an element which is invertible in  $C_i$ . Then, by assumption, there is an element

$$y = y_1 + y_2 \in \mathbb{K} \oplus C_{S,T} = C_i$$

with

$$\begin{aligned} (x_1 + x_2)(y_1 + y_2) &= 1, \\ (y_1 + y_2)(x_1 + x_2) &= 1. \end{aligned}$$

This implies that  $x_2 y_2 = 1$  and  $y_2 x_2 = 1$ , because  $\mathbb{K}$  and  $\mathcal{T}$  are 2-sided ideals in  $C_i$ . Hence,  $x_2$  is invertible in  $C_{S,T}$  and

$$\begin{aligned} (x_2^{-1} x_1 + 1)(x_2 y_1 + 1) &= 1, \\ (x_2 y_1 + 1)(x_2^{-1} x_1 + 1) &= 1. \end{aligned}$$

Regarding  $\xi := x_2^{-1} x_1 + 1$  as an element in  $\mathcal{T}_+$ , the unitalization of  $\mathcal{T}$ , this means that  $\xi$  is invertible in  $\mathbb{K}_+$ . But because each element in  $\mathcal{T}_+$  that is invertible in  $\mathcal{K}_+$  is already invertible in  $\mathcal{T}_+$ , the element  $\xi$  is invertible in  $\mathcal{T}_+$  and therefore  $x_2 y_1 + 1 \in \mathcal{T}_+$ . We conclude  $y_1 + y_2 \in D$ .

The same argument applies to matrices over  $D$ .  $\square$

This lemma implies that the inclusion

$$D \rightarrow C_i$$

induces an isomorphism after applying the  $K$ -theory functor (see [1], III.5). The trace  $\tau : D \rightarrow \mathbb{C}$  induces a group homomorphism  $K_0(D) \rightarrow \mathbb{Z}$  that we also denote by  $\tau$ . Because the  $K$ -theories of  $D$  and  $C_i$  coincide, we may consider  $\tau$  as being defined on  $K_0(C_i)$ . The restriction of this homomorphism to the canonical  $\mathbb{Z}$ -summand in  $K_0(C_i)$  is the identity.

Each of the transition functions  $\rho_{\lambda\kappa}$  takes values in the unitaries of  $q_i D$ . Hence, similarly to the definition of  $F_i$ , we get a smooth Hilbert  $D$ -module bundle  $\mathcal{W} \rightarrow M$  with connection  $\nabla_{\mathcal{W}}$ .

We are now in a situation where Theorem 9.2 of [9] can be applied. In our case it says that (up to sign)

$$z_i = \tau(\text{ind}(D_{F_i})) = \int_M \mathcal{A}(M) \cup \text{ch}_\tau(\mathcal{W}, \nabla_{\mathcal{W}}).$$

For the definition of  $\text{ch}_\tau$ , we refer to [9], Definition 5.1.

*Remark 2.5.* The number  $z_i$  can be determined without Lemma 2.4 by calculating first  $\tau(\text{ind}(D_{\mathcal{W}}))$  and applying a diagram chase comparing the split exact  $K$ -theory sequences for the algebras  $D$  and  $C_i$ . However, the first step requires an appropriate (yet unproblematic) reformulation of the Chern-Weil calculus of [9]. The above argument avoids this and fits more neatly the statement of Theorem 9.2 in [9].

Let us calculate the right hand side over an open subset  $U_\kappa \subset M$ . Using the trivialization of  $F|_{f^{-1}(U_\kappa)}$  from above and the definition of the trace  $\tau$ , we have

$$\int_{U_\kappa} \mathcal{A}(M) \cup \text{ch}_\tau(\mathcal{W}, \nabla_{\mathcal{W}}) = \int_{f^{-1}(U_\kappa)} f^*(\mathcal{A}(M)) \cup (\text{ch}(F - \underline{\mathbb{C}}^d, \nabla_F - \nabla_{\mathbb{C}^d})).$$

Note that the right hand integral involves only finitely many components of  $f^{-1}(U_\kappa)$ . Hence, by use of a partition of unity subordinate to the cover  $(U_\kappa)$  of  $M$ , we get

$$z_i = \tau(\text{ind}(D_{F_i})) = \int_{\overline{M}} \mathcal{A}(\overline{M}) \cup (\text{ch}(F - \underline{\mathbb{C}}^d, \nabla_F - \nabla_{\mathbb{C}^d})).$$

By construction,  $\text{ch}(F - \underline{\mathbb{C}}^d)$  represents a compactly supported nonzero cohomology class of degree  $n$  in  $\overline{M}$ . This implies  $z_i \neq 0$  and completes the proof of Proposition 1.5.

Note that, by the relative index formula of [3],  $z_i$  is exactly the relative index of Dirac operator on  $\overline{M}$  twisted on the one hand with the trivial bundle of rank  $d$ , and on the other hand twisted with  $F$ . In retrospective, we gave a definition of the relative index using suitable  $C^*$ -algebras and their  $K$ -theory (as in [6]) and reduced the proof of the corresponding index formula to the methods described in [9].

## REFERENCES

- [1] B. Blackadar, *K-theory for operator algebras*, Second edition, Cambridge University Press, Cambridge, 1998. 2
- [2] B. Blackadar, *Operator algebras*, Encyclopedia of mathematical sciences **122**, Springer Verlag (2006). 2
- [3] M. Gromov, H. B. Lawson, *Spin and Scalar curvature in the presence of a fundamental group I*, Ann. of Math.(2) **111** (1980), 209-230. (document), 1, 1, 2
- [4] M. Gromov, H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. IHES **58** (1983), 83-196. (document), 1, 1

- [5] B. Hanke, T. Schick, *Enlargeability and index theory*, J. Differential Geom. **74** (2) (2006), 293-320. (document), 1, 1.1, 1, 1, 1, 1
- [6] J. Roe, *A note on the relative index theorem*, Quart. J. Math. Oxford Ser. (2) **42** (1991), 365-373. 2.1, 2
- [7] J. Rosenberg, *Positive scalar curvature and the Novikov conjecture III*, Topology **25** (1986), 319-336. 1
- [8] T. Schick, *A counterexample to the (unstable) Gromov–Lawson–Rosenberg conjecture*, Topology **37** (1998), 1165–1168. 1
- [9] T. Schick,  *$L^2$ -index theorems,  $KK$ -theory and connections*, New York J. Math. **11** (2005), 387-443. 1, 2, 2, 2.5, 2
- [10] S. Stolz, *Manifolds of positive scalar curvature*, in: T. Farrell et al. (eds.), *Topology of high dimensional manifolds* (Trieste 2001), ICTP Lect. Notes vol. **9**, 661-709. 1

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN, GERMANY,  
*E-mail address:* `hanke@uni-math.gwdg.de`

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN, GERMANY,  
*E-mail address:* `schick@uni-math.gwdg.de`